

On ground fields of arithmetic hyperbolic reflection groups. II

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Abstract

This paper continues [18] (arXiv.org:math.AG/0609256) and [19] (arXiv:0708.3991).

Using authors's methods of 1980, 1981, some explicit finite sets of number fields containing all ground fields of arithmetic hyperbolic reflection groups in dimensions at least 4 are defined, and good explicit bounds of their degrees (over \mathbb{Q}) are obtained. This could be important for further classification.

Thus, now, an explicit bound of degree of ground fields of arithmetic hyperbolic reflection groups is unknown in dimension 3 only.

To 70th Birthday of Érnst Borisovich Vinberg

1 Introduction

This paper continues [19]. See the introduction of this paper about history, definitions and results concerning the subject.

In [19] some explicit finite sets of totally real algebraic number fields containing all ground fields of arithmetic hyperbolic reflection groups in dimensions $n \geq 6$ were defined, and good explicit bounds of degrees (over \mathbb{Q}) of their fields were obtained. In particular, an explicit bound (≤ 56) of degree of the ground field of any arithmetic hyperbolic reflection group in dimension $n \geq 6$ was obtained.

Here we continue this study for smaller dimensions $n = 5$ and 4. Using similar methods, we define some explicit finite sets of totally real algebraic number fields containing all ground fields of arithmetic hyperbolic reflection groups in dimensions $n \geq 4$. Moreover, an explicit bound (≤ 138) of degrees of fields from these sets are obtained. This requires much more difficult considerations comparing to $n \geq 6$, and it is very surprising to the author that this can be

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done. Thus, degree of the ground field of any arithmetic hyperbolic reflection group of dimension $n \geq 4$ is bounded by 138.

It is also very important that all these fields are attached to fundamental chambers of arithmetic hyperbolic reflection groups, and they can be further geometrically investigated and restricted.

It was also shown in [19] (using results of [13] and [4], [23]) that degree of the ground field of any arithmetic hyperbolic reflection group of dimension $n = 2$ is bounded by 44. Thus, an explicit bound of degree of ground fields of arithmetic hyperbolic reflection groups remains unknown in dimension $n = 3$ only. Finiteness of the number of maximal arithmetic hyperbolic reflection groups (and then a theoretical existence of this bound) was shown by Agol [1].

Since this paper is a direct continuation of [19], we use notations, definitions and results of this paper without their reminding.

In [26] (1981) and [27], Érnst Borisovich Vinberg gave a very short list of all possible ground fields (thirteen fields) of arithmetic hyperbolic reflection groups in dimensions $n \geq 14$. Thus, the results of [19] and of this paper can be vieweded as some extension of these beautiful Vinberg's results to smaller dimensions.

2 Ground fields of arithmetic hyperbolic reflection groups in dimensions $n \geq 4$

Since this paper is a direct continuation of [19], we use notations, definitions and results of this paper without their reminding.

In [19, Secs 3 and 4], explicit finite sets \mathcal{FL}^4 , \mathcal{FT} , $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$, and $\mathcal{F}\Gamma_{2,4}(14)$ of totally real algebraic number fields were defined. The set \mathcal{FL}^4 consists of all ground fields of arithmetic Lannér diagrams with ≥ 4 vertices and consists of three fields of degree ≤ 2 . The set \mathcal{FT} consists of all ground fields of arithmetic triangles (plane) and has 13 fields of degree ≤ 5 (it includes \mathcal{FL}^4). The set $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$, consists of all ground fields of V-arithmetic edge polyhedra of minimality 14 with connected Gram graph having 4 vertices. They are determined by 5 types of graphs $\Gamma_i^{(4)}(14)$, $i = 1, 2, 3, 4, 5$. The degrees of fields from these sets are bounded by 22, 39, 53, 56, 54 respectively. The set $\mathcal{F}\Gamma_{2,4}(14)$ consists of all ground fields of arithmetic quadrangles (plane) of minimality 14. Their degrees are bounded by 11.

The following result was obtained in [19, Theorem 4.5] using methods of [14] and [15] and results by Borel [4] and Takeuchi [23].

Theorem 2.1. *([19]) In dimensions $n \geq 6$, the ground field of any arithmetic hyperbolic reflection group belongs to one of finite sets of fields \mathcal{FL}^4 , \mathcal{FT} , $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$, and $\mathcal{F}\Gamma_{2,4}(14)$. In particular, its degree is bounded by 56.*

Applying the same methods and similar, but much more difficult considerations, here we want to extend this result to $n \geq 4$, also considering $n = 5$ and $n = 4$.

First, we introduce some other explicit finite sets of fields. All of them are related to fundamental pentagons on hyperbolic plane. Similarly \mathcal{FT} is related to arithmetic triangles on hyperbolic plane, and $\mathcal{F}\Gamma_{2,4}(14)$ is related to arithmetic quadrangles on hyperbolic plane.

Let us consider plane (or Fuchsian) arithmetic hyperbolic reflection groups W with a pentagon fundamental polygon Δ of minimality 14. We remind that this means that the set $P(\Delta) = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ of all perpendicular with square (-2) and directed outwards vectors to codimension one faces of Δ satisfies the condition

$$\delta_i \cdot \delta_j < 14, \forall \delta_i, \delta_j \in P(\Delta).$$

Respectively, we call Δ as an *arithmetic pentagon of minimality 14*.

Definition 2.2. We denote by $\Gamma_{2,5}(14)$ the set of Gram graphs $\Gamma(P(\Delta))$ of all arithmetic pentagons Δ of minimality 14. The set $\mathcal{F}\Gamma_{2,5}(14)$ consists of all their ground fields.

By Borel [4] and Takeuchi [23], for fixed $g \geq 0$ and $t \geq 0$, the number of arithmetic Fuchsian groups with signatures $(g; e_1, e_2, \dots, e_t)$ is finite. Applying this result to $g = 0$ and $t = 5$, we obtain that sets of arithmetic quadrangles $\Gamma_{2,5}$ and their ground fields $\mathcal{F}\Gamma_{2,5}$ are finite. Then their subsets $\Gamma_{2,5}(14)$ and $\mathcal{F}\Gamma_{2,5}(14)$ are also finite.

Moreover, in [23, pages 383–384] an upper bound n_0 of the degree of ground fields of Fuchsian groups with signatures $(g; e_1, e_2, \dots, e_t)$ is given. It is

$$n_0 = (b + \log_e C(g, t)) / \log_e (a / (2\pi)^{4/3}) \quad (1)$$

where

$$a = 29.099, \quad b = 8.3185, \quad C(g, t) = 2^{2g+t-2} (2g + t - 2)^{2/3}$$

(here a and b are due to Odlyzko). It follows that

$$[\mathbb{K} : \mathbb{Q}] \leq 12 \quad \text{for } \mathbb{K} \in \mathcal{F}\Gamma_{2,5} \supset \mathcal{F}\Gamma_{2,5}(14). \quad (2)$$

Let us consider V-arithmetic 3-dimensional chambers which are defined by the Gram graphs $\Gamma_1^{(6)}$ with 6 vertices $\delta_1, \dots, \delta_5, e$ shown in Figure 1. It follows that the corresponding V-arithmetic chamber \mathcal{M} satisfies the following condition: the 2-dimensional face \mathcal{M}_e of \mathcal{M} which is perpendicular to e is a pentagon Δ where

$$P(\Delta) = \{\delta_1, \delta_2, \delta_3, \tilde{\delta}_4, \delta_5\}$$

for

$$\tilde{\delta}_4 = \frac{\delta_4 + \cos(\pi/m)e}{\sin(\pi/m)},$$

are perpendicular to 5 consecutive sides of the pentagon. It has one angle π/k (defined by δ_1, δ_2) and all other its angles are right ($= \pi/2$). Thus, all planes \mathcal{H}_{δ_i} , $1 \leq i \leq 5$, are perpendicular to \mathcal{H}_e except \mathcal{H}_{δ_4} which has angle π/m , $m \geq 3$, with the plane \mathcal{H}_e .

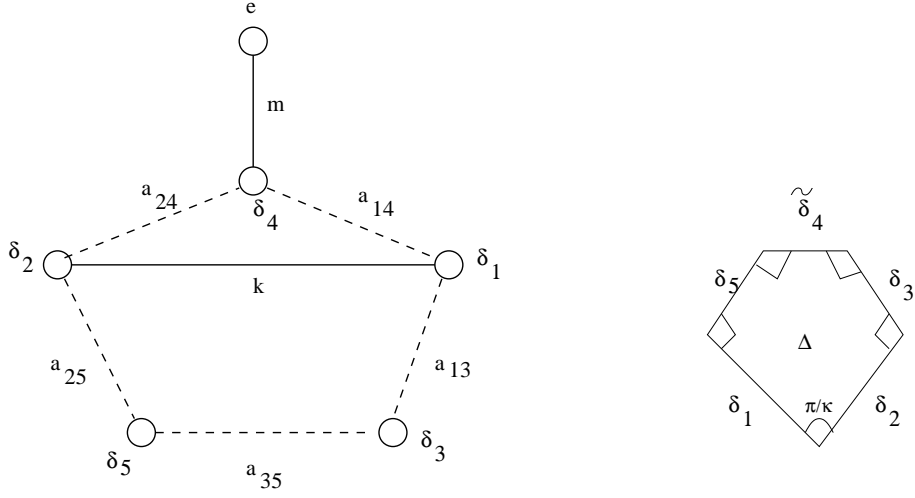


Figure 1: The pentagon graphs $\Gamma_1^{(6)}$

Definition 2.3. We denote by $\Gamma_1^{(6)}(14)$ the set of all V-arithmetic 3-dimensional graphs $\Gamma_1^{(6)}$ (or the corresponding 3-dimensional V-arithmetic chambers) of minimality 14. Thus, inequalities $2 < a_{ij} < 14$ satisfy. We denote by $\mathcal{F}\Gamma_1^{(6)}(14)$ the set of all their ground fields.

Similarly, we define V-arithmetic graphs $\Gamma_2^{(6)}$, $\Gamma_3^{(6)}$ given in Figure 2 which must (by definition) define 3-dimensional V-arithmetic chambers \mathcal{M} . For both of them, the face \mathcal{M}_e of \mathcal{M} which is perpendicular to e , is a pentagon Δ with right angles. For $\Gamma_2^{(6)}$,

$$P(\Delta) = \{\tilde{\delta}_1, \delta_2, \delta_3, \delta_4, \delta_5\}$$

correspond to five consecutive sides of Δ where

$$\tilde{\delta}_1 = \frac{\delta_1 + \cos(\pi/m)e}{\sin(\pi/m)}.$$

For $\Gamma_3^{(6)}$,

$$P(\Delta) = \{\tilde{\delta}_1, \delta_2, \tilde{\delta}_3, \delta_4, \delta_5\}$$

correspond to five consecutive sides of Δ where

$$\tilde{\delta}_1 = \frac{\delta_1 + \cos(\pi/m_1)e}{\sin(\pi/m_1)}, \quad \tilde{\delta}_3 = \frac{\delta_3 + \cos(\pi/m_3)e}{\sin(\pi/m_3)}.$$

V-arithmetic graphs $\Gamma_1^{(7)}$ and $\Gamma_2^{(7)}$ of Figure 2 must (by definition) define 4-dimensional V-arithmetic fundamental chambers \mathcal{M} . For both of them, the

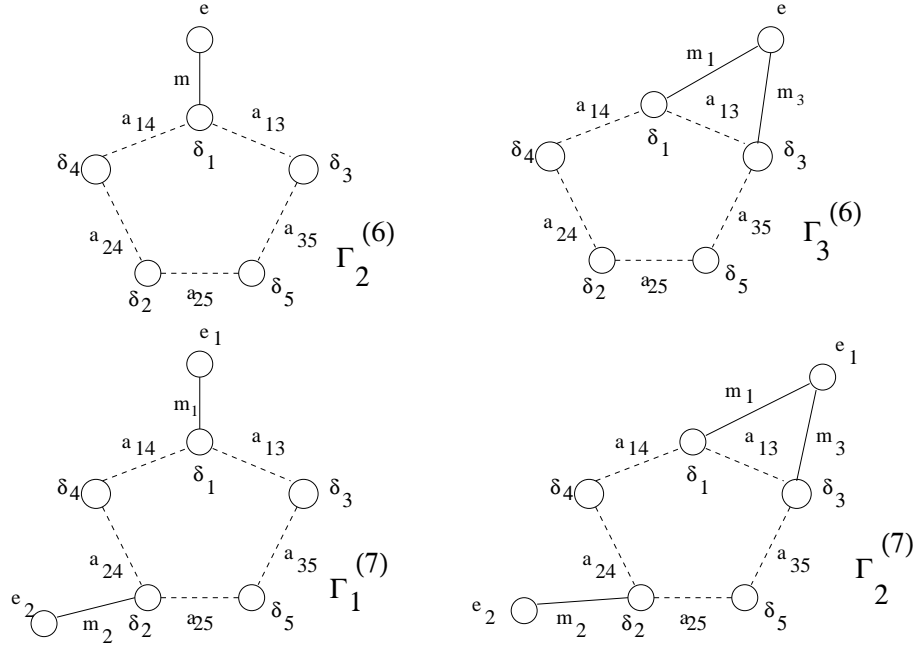


Figure 2: The pentagon graphs $\Gamma_2^{(6)}$, $\Gamma_3^{(6)}$, $\Gamma_1^{(7)}$, $\Gamma_2^{(7)}$

2-dimensional face \mathcal{M}_{e_1, e_2} of \mathcal{M} which is perpendicular to both e_1 and e_2 , is a pentagon Δ with right angles. For $\Gamma_1^{(7)}$,

$$P(\Delta) = \{\tilde{\delta}_1, \tilde{\delta}_2, \delta_3, \delta_4, \delta_5\}$$

correspond to five consecutive sides of Δ where

$$\tilde{\delta}_1 = \frac{\delta_1 + \cos(\pi/m_1)e_1}{\sin(\pi/m_1)}, \quad \tilde{\delta}_2 = \frac{\delta_2 + \cos(\pi/m_2)e_2}{\sin(\pi/m_2)}.$$

For $\Gamma_2^{(7)}$,

$$P(\Delta) = \{\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3, \delta_4, \delta_5\}$$

correspond to five consecutive sides of Δ where

$$\tilde{\delta}_1 = \frac{\delta_1 + \cos(\pi/m_1)e_1}{\sin(\pi/m_1)}, \quad \tilde{\delta}_2 = \frac{\delta_2 + \cos(\pi/m_2)e_2}{\sin(\pi/m_2)}, \quad \tilde{\delta}_3 = \frac{\delta_3 + \cos(\pi/m_3)e_1}{\sin(\pi/m_3)}.$$

Definition 2.4. We denote by $\Gamma_2^{(6)}(14)$, $\Gamma_3^{(6)}(14)$, $\Gamma_1^{(7)}(14)$ and $\Gamma_2^{(7)}(14)$ the sets of all V -arithmetic graphs $\Gamma_2^{(6)}$, $\Gamma_3^{(6)}$, $\Gamma_1^{(7)}$ and $\Gamma_2^{(7)}$ respectively (or the corresponding 3-dimensional or 4-dimensional V -arithmetic fundamental chambers) of minimality 14. Thus, inequalities $2 < a_{ij} < 14$ satisfy. We denote

by $\mathcal{F}\Gamma_2^{(6)}(14)$, $\mathcal{F}\Gamma_3^{(6)}(14)$, $\mathcal{F}\Gamma_1^{(7)}(14)$ and $\mathcal{F}\Gamma_2^{(7)}(14)$ the sets of all their ground fields respectively.

We have

Theorem 2.5. *The sets of V -arithmetic graphs $\Gamma_1^{(6)}(14)$, $\Gamma_2^{(6)}(14)$, $\Gamma_3^{(6)}(14)$, $\Gamma_1^{(7)}(14)$, $\Gamma_2^{(7)}(14)$ and their fields $\mathcal{F}\Gamma_1^{(6)}(14)$, $\mathcal{F}\Gamma_2^{(6)}(14)$, $\mathcal{F}\Gamma_3^{(6)}(14)$, $\mathcal{F}\Gamma_1^{(7)}(14)$, $\mathcal{F}\Gamma_2^{(7)}(14)$ are finite.*

Degree of any field from $\mathcal{F}\Gamma_1^{(6)}(14)$ is bounded (\leq) by 56.

Degree of any field from $\mathcal{F}\Gamma_2^{(6)}(14)$ is bounded by 75.

Degree of any field from $\mathcal{F}\Gamma_3^{(6)}(14)$ is bounded by 138.

Degree of any field from $\mathcal{F}\Gamma_1^{(7)}(14)$ is bounded by 42.

Degree of any field from $\mathcal{F}\Gamma_2^{(7)}(14)$ is bounded by 138.

Proof. The proof requires long considerations and calculations. It will be given in a special Sect. 3. \square

We have the following main result of the paper.

Theorem 2.6. *In dimensions $n \geq 4$, the ground field of any arithmetic hyperbolic reflection group belongs to one of finite sets of fields $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$, $\mathcal{F}\Gamma_{2,4}(14)$ and $\mathcal{F}\Gamma_1^{(6)}(14)$, $\mathcal{F}\Gamma_2^{(6)}(14)$, $\mathcal{F}\Gamma_3^{(6)}(14)$, $\mathcal{F}\Gamma_1^{(7)}(14)$, $\mathcal{F}\Gamma_2^{(7)}(14)$, $\mathcal{F}\Gamma_{2,5}(14)$.*

In particular, its degree is bounded by 138.

Proof. By [19], if $n \geq 6$, the ground field \mathbb{K} belongs to one of sets $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$, $\mathcal{F}\Gamma_{2,4}(14)$. Thus, further we can assume that the ground field \mathbb{K} does not belong to these sets, and the dimension is equal to $n = 4$ or $n = 5$.

Let W be an arithmetic hyperbolic reflection group of dimension $n = 4$ or 5, \mathcal{M} is its fundamental chamber, and $P(\mathcal{M})$ is the set of all vectors with square (-2) which are perpendicular to codimension one faces of \mathcal{M} and directed outwards. For $\delta \in P(\mathcal{M})$ we denote by \mathcal{H}_δ and \mathcal{M}_δ the hyperplane and the codimension one face $\mathcal{M} \cap \mathcal{H}_\delta$ respectively which is perpendicular to δ .

By [14], there exists $e \in P(\mathcal{M})$ which defines a narrow face \mathcal{M}_e of \mathcal{M} of minimality 14. It means that $\delta_1 \cdot \delta_2 < 14$ for any $\delta_1, \delta_2 \in P(\mathcal{M}, e) \subset P(\mathcal{M})$. Here

$$P(\mathcal{M}, e) = \{\delta \in P(\mathcal{M}) \mid \mathcal{H}_\delta \cap \mathcal{H}_e \neq \emptyset\}.$$

The field \mathbb{K} is different from \mathbb{Q} . It is well-known that then \mathcal{M} is compact, and it is simple: any vertex of \mathcal{M} is contained in exactly n codimension one faces. Then \mathcal{M}_e is also simple $n - 1$ -dimensional polyhedron. If $n = 5$, it is 4-dimensional, and if $n = 4$, it is 3-dimensional.

By formula [19, (7)] which was proved in [15], the average number $\alpha_{n-1}^{(0,2)}$ of vertices of 2-dimensional faces of any simple $n - 1$ -dimensional polyhedron

satisfies for $n \geq 4$ the inequality

$$\alpha_{n-1}^{(0,2)} < 4 + \begin{cases} \frac{4}{n-2} & \text{if } n \text{ is even,} \\ \frac{4}{n-3} & \text{if } n \text{ is odd.} \end{cases} \quad (3)$$

It follows that $\alpha_{n-1}^{(0,2)} < 6$, and \mathcal{M}_e has a 2-dimensional face which is a polygon Δ with ≤ 5 vertices. Remark that the same can be obtained by Euler's formula for Euler characteristic $b_0 - b_1 + b_2 = 2$ applied to any 3-dimensional face of \mathcal{M}_e . Thus, for $n = 4$ and $n = 5$, actually, we don't need the inequality (3).

If Δ is a triangle, then obviously \mathbb{K} belongs to \mathcal{FL}^4 or \mathcal{FT} (see [19, Lemma 4.6]). If Δ is a quadrangle and \mathbb{K} is different from \mathcal{FL}^4 , \mathcal{FT} and $\mathcal{FT}_i^{(4)}(14)$, $1 \leq i \leq 5$, then \mathbb{K} belongs to $\mathcal{FT}_{2,4}(14)$, see [19, Lemma 4.7]. Thus, further, we can assume that Δ is a pentagon.

By [19, Lemma 4.3], if \mathbb{K} does not belong to \mathcal{FL}^4 , \mathcal{FT} and $\mathcal{FT}_i^{(4)}(14)$, $1 \leq i \leq 4$, then the Coxeter graph $C(v)$ of any vertex $v \in \mathcal{M}_e$ has all connected components having only one or two vertices. If additionally \mathbb{K} does not belong to $\mathcal{FT}_5^{(5)}(14)$, then the hyperbolic connected component of the edge graph $\Gamma(r)$ defined by any edge $r = v_1 v_2 \subset \mathcal{M}_e$ has ≤ 3 vertices.

By direct check (see the proof of [19, Lemma 4.7]), we see that then there are only 3 cases where we denote by $Q \subset P(\mathcal{M})$ the set of all perpendicular to Δ elements $\delta \in P(\mathcal{M})$, and by $\delta_i \in P(\mathcal{M}) - Q$, $i = 1, 2, 3, 4, 5$, perpendicular vectors to five consecutive edges of Δ . Thus, Q consists of $n - 2$ elements, and the plane of Δ is intersection of hyperplanes \mathcal{H}_δ , $\delta \in Q$. Moreover, $\mathcal{H}_{\delta_i} \cap \Delta$, $i = 1, 2, 3, 4, 5$, give lines of five consecutive edges of Δ .

Case 1. $Q \perp \delta_i$, $1 \leq i \leq 5$. Then Δ is a fundamental pentagon of minimality 14 for arithmetic hyperbolic reflection group with $P(\Delta) = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$. Then \mathbb{K} is the ground field of this group and $\mathbb{K} \in \mathcal{FT}_{2,5}(14)$.

Case 2. Q is not perpendicular to all δ_i , $1 \leq i \leq 5$, and Δ has a non-right angle. We can assume that the vertex of this angle is perpendicular to δ_1 and δ_2 . Thus, $\delta_1 \cdot \delta_2 = 2 \cos(\pi/k)$, $k \geq 3$. Then all other angles of Δ are $\pi/2$. There exists exactly one element $e \in Q$ which is not perpendicular all δ_i , $1 \leq i \leq 5$. Then e is perpendicular to all δ_i except δ_4 , and $e \cdot \delta_4 = 2 \cos(\pi/m)$, $m \geq 3$. Thus, e and δ_i , $1 \leq i \leq 5$, define the hyperbolic graph $\Gamma_1^{(6)}(14)$, and \mathbb{K} is the ground field of this hyperbolic graph. (This case is very similar to [19, Lemma 4.7].)

Case 3. Q is not perpendicular to all δ_i , $1 \leq i \leq 5$, but all angles of Δ are right. Then δ_i , $1 \leq i \leq 5$, and elements from Q which are not perpendicular to all these elements define one of hyperbolic graphs $\Gamma_2^{(6)}(14)$, $\Gamma_3^{(6)}(14)$, $\Gamma_1^{(7)}(14)$, $\Gamma_2^{(7)}(14)$. The field \mathbb{K} is the ground field of one of these hyperbolic graphs.

We leave the corresponding routine and simple check to a reader.

This finishes the proof of the theorem. \square

3 V-arithmetic pentagon graphs $\Gamma_1^{(6)}(14), \Gamma_2^{(6)}(14), \Gamma_3^{(6)}(14), \Gamma_1^{(7)}(14), \Gamma_2^{(7)}(14)$ and their fields

Here we prove Theorem 2.5.

3.1 Some general results.

We use the following general results from [15].

Theorem 3.1. ([15, Theorem 1.2.1]) *Let \mathbb{F} be a totally real algebraic number field, and let each embedding $\sigma : \mathbb{F} \rightarrow \mathbb{R}$ corresponds to an interval $[a_\sigma, b_\sigma]$ in \mathbb{R} where*

$$\prod_{\sigma} \frac{b_{\sigma} - a_{\sigma}}{4} < 1.$$

In addition, let the natural number m and the intervals $[s_1, t_1], \dots, [s_m, t_m]$ in \mathbb{R} be fixed. Then there exists a constant $N(s_i, t_i)$ such that, if α is a totally real algebraic integer and if the following inequalities hold for the embeddings $\tau : \mathbb{F}(\alpha) \rightarrow \mathbb{R}$:

$$s_i \leq \tau(\alpha) \leq t_i \quad \text{for } \tau = \tau_1, \dots, \tau_m,$$

$$a_{\tau|\mathbb{F}} \leq \tau(\alpha) \leq b_{\tau|\mathbb{F}} \quad \text{for } \tau \neq \tau_1, \dots, \tau_m,$$

then

$$[\mathbb{F}(\alpha) : \mathbb{F}] \leq N(s_i, t_i).$$

Theorem 3.2. ([15, Theorem 1.2.2]) *Under the conditions of Theorem 3.1, $N(s_i, t_i)$ can be taken to be $N(s_i, t_i) = N_0$, where N_0 is the least natural number solution of the inequality*

$$N_0 M \ln(1/R) - M \ln(N_0 + 1) - \ln B \geq \ln S. \quad (4)$$

Here

$$M = [\mathbb{F} : \mathbb{Q}], \quad B = 2\sqrt{|\text{discr } \mathbb{F}|}; \quad (5)$$

$$R = \sqrt{\prod_{\sigma} \frac{b_{\sigma} - a_{\sigma}}{4}}, \quad S = \prod_{i=1}^m \frac{2er_i}{b_{\sigma_i} - a_{\sigma_i}} \quad (6)$$

where

$$\sigma_i = \tau_i|_{\mathbb{F}}, \quad r_i = \max\{|b_i - a_{\sigma_i}|, |b_{\sigma_i} - a_i|\}. \quad (7)$$

We note that the proof of Theorems 3.1 and 3.2 uses a variant of Fekete's Theorem (1923) about existence of non-zero integer polynomials of bounded degree which differ only slightly from zero on appropriate intervals. See [15, Theorem 1.1.1].

Below we will apply these results in two cases which are very similar to used in [19, Sec. 5.5] and [15].

Case 1. For a natural $l \geq 3$ we denote $\mathbb{F}_l = \mathbb{Q}(\cos(2\pi/l))$. We consider a totally real algebraic number field \mathbb{K} where $\mathbb{F}_l \subset \mathbb{K} = \mathbb{Q}(\alpha)$, and the algebraic integer α satisfies

$$0 < \sigma(\alpha) < a\sigma(\sin^2(\pi/l)) \quad (8)$$

for all $\sigma : \mathbb{K} \rightarrow \mathbb{R}$ such that $\sigma \neq \sigma^{(+)}$, and

$$b_1 < \sigma^{(+)}(\alpha) < b_2 \quad (9)$$

where $\sigma^{(+)} : \mathbb{K} \rightarrow \mathbb{R}$ is the identity. We assume that $0 < a < 4$ and $b_1 < b_2$. We denote $b = \max\{|b_1|, |b_2|\}$, and we assume that $a \leq b$. We want to estimate $[\mathbb{K} : \mathbb{F}_l] = N_0$ and $N = [\mathbb{K} : \mathbb{Q}] = N_0[\mathbb{F}_l : \mathbb{Q}]$ from above.

For $l \geq 3$, we have $[\mathbb{F}_l : \mathbb{Q}] = \varphi(l)/2$ where $\varphi(l)$ is the Euler function, and $N_{\mathbb{F}_l/\mathbb{Q}}(\sin^2(\pi/l)) = \gamma(l)/4^{\varphi(l)/2}$ where

$$N_{\mathbb{F}_l/\mathbb{Q}}(4\sin^2(\pi/l)) = \gamma(l) = \begin{cases} p & \text{if } l = p^t > 2 \text{ where } p \text{ is prime,} \\ 1 & \text{otherwise.} \end{cases} \quad (10)$$

We have

$$\frac{ba^N |N_{\mathbb{K}/\mathbb{Q}}(\sin^2(\pi/l))|}{a \sin^2(\pi/l)} > |N_{\mathbb{K}/\mathbb{Q}}(\alpha)| \geq 1$$

and

$$\frac{b(a/4)^N \gamma(l)^{2N/\varphi(l)}}{a \sin^2(\pi/l)} > 1.$$

Equivalently, we have

$$N \left(\ln \frac{2}{\sqrt{a}} - \frac{\ln \gamma(l)}{\varphi(l)} \right) < \ln \sqrt{\frac{b}{a}} - \ln \sin \frac{\pi}{l} \quad \text{and} \quad (\varphi(l)/2) | N. \quad (11)$$

Since $\gamma(l) \leq l$, $\varphi(l) \geq Cl/\ln(\ln l)$ for $l \geq 6$ where $C = \varphi(6) \ln(\ln 6)/6 \geq 0.194399$, $\sin(\pi/l) \leq \pi/l$ for $l \geq 3$, there exists only finite number of $l \geq 3$ such that (11) has solutions $N \in \mathbb{N}$.

More exactly, there exists only finite number of *exceptional* $l \geq 3$ such that

$$\ln \frac{2}{\sqrt{a}} - \frac{\ln \gamma(l)}{\varphi(l)} \leq 0. \quad (12)$$

All non-exceptional l satisfy the inequality

$$(\varphi(l)/2) \left(\ln \frac{2}{\sqrt{a}} - \frac{\ln \gamma(l)}{\varphi(l)} \right) < \ln \sqrt{\frac{b}{a}} - \ln \sin \frac{\pi}{l}. \quad (13)$$

Remark that exceptional l also satisfy this inequality.

If $\gamma(l) = 1$, (13) implies that l satisfies the inequality

$$(C/2) \ln(2/\sqrt{a}) l < \left(\ln l + \ln(\sqrt{(b/a)/\pi}) \right) \ln \ln l. \quad (14)$$

It follows that

$$l < L_0 \quad (15)$$

where $L_0 > 3$ satisfies

$$(C/2) \ln(2/\sqrt{a}) L_0 \geq \left(\ln L_0 + \ln(\sqrt{(b/a)}/\pi) \right) \ln \ln L_0. \quad (16)$$

If $l = p^t$ where p is prime, (13) implies that l satisfies the inequality

$$(C/2) \Delta(a) l < \left(\ln l + \ln(\sqrt{(b/a)}/\pi) \right) \ln \ln l \quad (17)$$

where

$$\Delta(a) = \min_{l=p^t \geq L_0} \left\{ \ln \frac{2}{\sqrt{a}} - \frac{\ln \gamma(l)}{\varphi(l)} > 0 \right\}. \quad (18)$$

It follows that

$$l < L_1 \quad (19)$$

where $L_1 \geq L_0$ is a solution of the inequality

$$(C/2) \Delta(a) L_1 \geq \left(\ln L_1 + \ln(\sqrt{(b/a)}/\pi) \right) \ln \ln L_1. \quad (20)$$

Thus, to find all non-exceptional l satisfying (13), we should check (13) for all l such that $3 \leq l < L_1$, moreover, if $L_0 \leq l < L_1$, we can assume that $l = p^t$. Their number is finite, and all of them can be effectively found.

For non-exceptional l satisfying (13), we obtain bounds

$$N_0 = [\mathbb{K} : \mathbb{F}_l] \leq \left\lceil \frac{\ln \sqrt{b/a} - \ln \sin(\pi/l)}{(\varphi(l)/2) (\ln(2/\sqrt{a}) - (\ln \gamma(l))/\varphi(l))} \right\rceil \quad (21)$$

and

$$N = [\mathbb{K} : \mathbb{Q}] \leq \left\lceil \frac{\ln \sqrt{b/a} - \ln \sin(\pi/l)}{(\varphi(l)/2) (\ln(2/\sqrt{a}) - (\ln \gamma(l))/\varphi(l))} \right\rceil \cdot (\varphi(l)/2). \quad (22)$$

This using of the norm, we call the *Method B* (like in [19, Sec. 5.5]).

On the other hand, for fixed l , we obtain a bound for N_0 using Theorems 3.1, 3.2 applied to $\mathbb{F} = \mathbb{F}_l$ and α . We can take

$$R = \sqrt{|N_{\mathbb{F}_l/\mathbb{Q}}(\sin^2(\pi/l))|(a/4)^{\varphi(l)/2}} = \left(\frac{\gamma(l)^{1/\varphi(l)} a^{1/2}}{4} \right)^{\varphi(l)/2}, \quad (23)$$

where

$$R < 1 \text{ if and only if } \ln \frac{4}{\sqrt{a}} - \frac{\ln \gamma(l)}{\varphi(l)} > 0, \quad (24)$$

$$M = [\mathbb{F}_l : \mathbb{Q}] = \frac{\varphi(l)}{2}, \quad B = 2\sqrt{|\text{discr } \mathbb{F}_l|} \quad (25)$$

where the discriminant $|\text{discr } \mathbb{F}_l|$ is given in (93), and

$$S = \frac{2e \max\{a, b_2, a - b_1\}}{a \sin^2(\pi/l)}. \quad (26)$$

Then $[\mathbb{K} : \mathbb{F}_l] \leq n_0$ and $[\mathbb{K} : \mathbb{Q}] \leq n_0 \varphi(l)/2$ where n_0 is the least natural solution of the inequality (4)

$$n_0 M \ln(1/R) - M \ln(n_0 + 1) - \ln B \geq \ln S. \quad (27)$$

In particular, this gives a bound for $[\mathbb{K} : \mathbb{Q}]$ for exceptional l satisfying (24) and improves the bound (21) for N_0 when it is poor, which also improves the bound for $[\mathbb{K} : \mathbb{Q}]$. This using of Theorems 3.1, 3.2, we call the *Method A* (like in [19, Sec. 5.5]).

We shall apply these Methods A and B to $\Gamma_2^{(6)}(14)$ in Sect. 3.3.1.

Case 2. For natural $k \geq s \geq 3$, we denote $\mathbb{F}_{k,s} = \mathbb{Q}(\cos(2\pi/k), \cos(2\pi/s))$. We consider a totally real algebraic number field \mathbb{K} where $\mathbb{F}_{k,s} \subset \mathbb{K} = \mathbb{Q}(\alpha)$, and the algebraic integer α satisfies

$$0 < \sigma(\alpha) < a \sigma(\sin^2(\pi/k) \sin^2(\pi/s)) \quad (28)$$

for all $\sigma : \mathbb{K} \rightarrow \mathbb{R}$ such that $\sigma \neq \sigma^{(+)}$, and

$$b_1 < \sigma^{(+)}(\alpha) < b_2 \quad (29)$$

where $\sigma^{(+)} : \mathbb{K} \rightarrow \mathbb{R}$ is the identity. We assume that $0 < a < 16$ and $b_1 < b_2$. We denote $b = \max\{|b_1|, |b_2|\}$, and we assume that $a \leq b$. We want to estimate $[\mathbb{K} : \mathbb{F}_{k,s}] = N_0$ and $N = [\mathbb{K} : \mathbb{Q}] = N_0 [\mathbb{F}_{k,s} : \mathbb{Q}]$ for non-exceptional k and s where $l \geq 3$ is called *exceptional* if

$$\ln \frac{4}{\sqrt{a}} - \frac{\ln \gamma(l)}{\varphi(l)} \leq 0. \quad (30)$$

Equivalently, we have $4/\sqrt{a} \leq \gamma(l)^{1/\varphi(l)}$. We also assume that $k \geq s \geq s_0 \geq 3$ where $s_0 \geq 3$ is fixed.

We have $[\mathbb{F}_{k,s} : \mathbb{Q}] = \varphi([k, s])/2\rho(k, s)$ where $\rho(k, s) = 1$ or 2 is given in (94), and $N_{\mathbb{F}_l/\mathbb{Q}}(\sin^2(\pi/l)) = \gamma(l)/4^{\varphi(l)/2}$ where $\gamma(l)$ is given in (10). We have

$$\frac{ba^N |N_{\mathbb{K}/\mathbb{Q}}(\sin^2(\pi/k))| |N_{\mathbb{K}/\mathbb{Q}}(\sin^2(\pi/s))|}{a \sin^2(\pi/k) \sin^2(\pi/s)} > |N_{\mathbb{K}/\mathbb{Q}}(\alpha)| \geq 1$$

and

$$\frac{b(a/16)^N \gamma(k)^{2N/\varphi(k)} \gamma(s)^{2N/\varphi(s)}}{a \sin^2(\pi/k) \sin^2(\pi/s)} > 1.$$

Equivalently, we obtain

$$N \left(\ln \frac{4}{\sqrt{a}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \right) < \ln \sqrt{\frac{b}{a}} - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s} \text{ and } \frac{\varphi([k, s])}{2\rho(k, s)} \mid N. \quad (31)$$

Since $\gamma(l) \leq l$, $\varphi(l) \geq Cl/\ln(\ln l)$ for $l \geq 6$ where $C = \varphi(6) \ln(\ln 6)/6$, $\sin(\pi/l) \leq \pi/l$ for $l \geq 3$, there exists only finite number of pairs (k, s) such that (31) has solutions $N \in \mathbb{N}$ where both k and s are non-exceptional.

More exactly, there exists only finite number of *exceptional pairs* (k, s) where a pair (k, s) (consisting of non-exceptional k and s) is called exceptional if

$$\ln \frac{4}{\sqrt{a}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \leq 0. \quad (32)$$

All non-exceptional pairs (k, s) satisfying (31) satisfy the inequality

$$\frac{\varphi([k, s])}{2\rho(k, s)} \cdot \left(\ln \frac{4}{\sqrt{a}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \right) < \ln \sqrt{\frac{b}{a}} - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s}. \quad (33)$$

Remark that exceptional pairs (k, s) also satisfy this inequality.

If $\gamma(k) = \gamma(s) = 1$ and $k \geq s$, (33) implies

$$(C/2) \ln(4/\sqrt{a})k < \left(2 \ln k + \ln(\sqrt{(b/a)}/\pi^2) \right) \ln \ln k. \quad (34)$$

It follows that

$$s_0 \leq s \leq k < K_0 \quad (35)$$

where $K_0 > 3$ satisfies

$$(C/2) \ln(4/\sqrt{a})K_0 \geq \left(2 \ln K_0 + \ln(\sqrt{(b/a)}/\pi^2) \right) \ln \ln K_0. \quad (36)$$

If one of $\gamma(k)$, $\gamma(s)$ is not equal to 1, then (33) implies for non-exceptional pairs (k, s) that

$$(C/2)\Delta_1(a)k < \left(2 \ln k + \ln(\sqrt{(b/a)}/\pi^2) \right) \ln \ln k \quad (37)$$

where

$$\Delta_1(a) = \min_{k \geq s \geq s_0, k \geq K_0} \left\{ \ln \frac{4}{\sqrt{a}} - \frac{\ln \gamma(s)}{\varphi(s)} - \frac{\ln \gamma(k)}{\varphi(k)} > 0 \right\}. \quad (38)$$

It follows that

$$s_0 \leq s \leq k < K_1 \quad (39)$$

where $K_1 \geq K_0$ is a solution of the inequality

$$(C/2)\Delta_1(a)K_1 \geq \left(2 \ln K_1 + \ln(\sqrt{(b/a)}/\pi^2) \right) \ln \ln K_1. \quad (40)$$

Thus, to find all non-exceptional pairs (k, s) satisfying (33), we should check (33) for all $s_0 \leq s \leq k < K_1$; moreover, if $K_0 \leq k \leq K_1$, we can assume that one of k and s is equal to p^t where p is prime. The number of such pairs is finite, and all of them can be effectively found.

For such non-exceptional pairs (k, s) satisfying (33), we obtain bounds

$$N_0 = [\mathbb{K} : \mathbb{F}_{k,s}] \leq \left\lceil \frac{\ln \sqrt{\frac{b}{a}} - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s}}{\frac{\varphi([k,s])}{2\rho(k,s)} \cdot \left(\ln \frac{4}{\sqrt{a}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \right)} \right\rceil \quad (41)$$

and

$$N = [\mathbb{K} : \mathbb{Q}] \leq \left[\frac{\ln \sqrt{\frac{b}{a}} - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s}}{\frac{\varphi([k,s])}{2\rho(k,s)} \cdot \left(\ln \frac{4}{\sqrt{a}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \right)} \right] \cdot \frac{\varphi([k,s])}{2\rho(k,s)}. \quad (42)$$

This using of the norm, we call the *Method B* (like in [19, Sec. 5.5]).

On the other hand, for a fixed pair (k, s) , we can obtain a bound for N_0 using Theorems 3.1, 3.2 applied to $\mathbb{F} = \mathbb{F}_{k,s}$ and α . We can take

$$R = \sqrt{|N_{\mathbb{F}_{k,s}/\mathbb{Q}}(\sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{s})| \left(\frac{a}{4} \right)^{\frac{\varphi([k,s])}{2\rho(k,s)}}} = \left(\frac{\gamma(k)^{\frac{1}{\varphi(k)}} \gamma(s)^{\frac{1}{\varphi(s)}} a^{\frac{1}{2}}}{8} \right)^{\frac{\varphi([k,s])}{2\rho(k,s)}} \quad (43)$$

where

$$R < 1 \text{ if and only if } \ln \frac{8}{\sqrt{a}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} > 0, \quad (44)$$

$$M = [\mathbb{F}_{k,s} : \mathbb{Q}] = \frac{\varphi([k,s])}{2\rho(k,s)}, \quad B = 2\sqrt{|\text{discr } \mathbb{F}_{k,s}|} \quad (45)$$

where the discriminant $|\text{discr } \mathbb{F}_{k,s}|$ is given in (96) and (97), and

$$S = \frac{2e \max\{a, b_2, a - b_1\}}{a \sin^2(\pi/s) \sin^2(\pi/k)}. \quad (46)$$

For all pairs (k, s) satisfying (44), we obtain the bounds $[\mathbb{K} : \mathbb{F}_{k,s}] \leq n_0$ and $[\mathbb{K} : \mathbb{Q}] \leq n_0 \varphi([k,s]) / (2\rho(k,s))$ where n_0 is the least natural solution of the inequality (4),

$$n_0 M \ln(1/R) - M \ln(n_0 + 1) - \ln B \geq \ln S.$$

For $a < 16$ and $k, s \geq 3$, all pairs (k, s) , except finite number, satisfy (44), and we can apply this method to all these pairs. In particular, this gives a bound for $[\mathbb{K} : \mathbb{Q}]$ for all exceptional pairs (k, s) satisfying (44), and it improves the bound (41) for N_0 when it is poor, which also improves the bound for $[\mathbb{K} : \mathbb{Q}]$. This using of Theorems 3.1, 3.2, we call the *Method A* (like in [19, Sec. 5.5]).

We apply these Methods A and B to $\Gamma_1^{(6)}(14)$ in Sect. 3.2 and $\Gamma_3^{(6)}(14)$, $\Gamma_1^{(7)}(14)$, $\Gamma_2^{(7)}(14)$ in Sect. 3.3.

3.2 V-arithmetic pentagon graphs $\Gamma_1^{(6)}(14)$ and their fields.

Here we consider V-arithmetic 3-dimensional graphs $\Gamma_1^{(6)}(14)$ and their fields. See Definition 2.3 and Figure 1.

First, let us consider a pentagon Δ on hyperbolic plane which has all angles right ($= \pi/2$) except one angle which is equal to π/k , $k \geq 2$. We denote $c = 2 \cos(\pi/k)$ where $-2 < c < 2$. When $c = 0$, we obtain a pentagon with right angles.

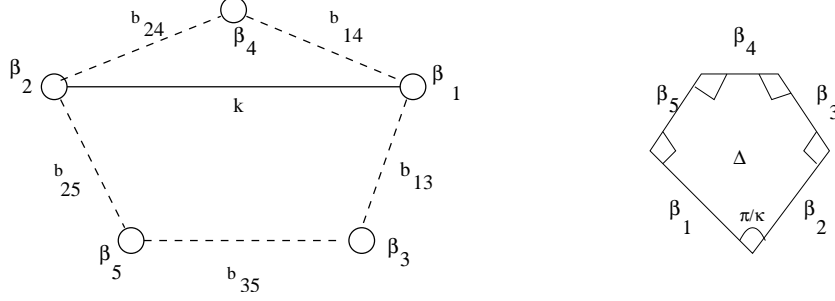


Figure 3: The graph of a pentagon with one non-right angle

Let

$$P(\Delta) = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$$

correspond to five consecutive sides of Δ and β_1, β_2 are perpendicular to the vertex of Δ with the angle π/k . Thus, $\beta_1 \cdot \beta_2 = c$ and $\beta_i \cdot \beta_{i+1} = 0$ for $2 \leq i \leq 5$. The Gram graph of Δ is given in Figure 3 where we denote $b_{ij} = \beta_i \cdot \beta_j$ with $b_{ij} > 2$.

The vectors $\beta_1, \beta_2, \beta_4$ generate the form Φ which defines the hyperbolic plane. The determinant of their Gram matrix is equal to

$$d_{124} = -8 + 2cb_{14}b_{24} + 2b_{14}^2 + 2b_{24}^2 + 2c^2. \quad (47)$$

It must be positive for the geometric embedding $\sigma^{(+)}$, and it must be negative for non-geometric embeddings $\sigma \neq \sigma^{(+)}$. Thus,

$$0 < b_{14}^2 + b_{24}^2 + cb_{14}b_{24} < 4 - c^2 \quad (48)$$

for $\sigma \neq \sigma^{(+)}$. Here the first inequality follows from $-2 < c < 2$ for any embedding since $c = 2\cos(\pi/k)$ where $k \geq 2$ is an integer.

Applying this to the pentagon Δ corresponding to $\Gamma_1^{(6)}$, we obtain $\beta_i = \delta_i$ for $i \neq 4$, and $\beta_4 = \tilde{\delta}_4 = (\delta_4 + \cos(\pi/m)e)/\sin(\pi/m)$. Then $b_{14} = a_{14}/\sin(\pi/m)$, $b_{24} = a_{24}/\sin(\pi/m)$ and for

$$\alpha = a_{14}^2 + a_{24}^2 + 2\cos(\pi/k)a_{14}a_{24} \quad (49)$$

we obtain

$$0 < \sigma(\alpha) < 4\sigma(\sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{m}) \quad (50)$$

for $\sigma \neq \sigma^{(+)}$, and

$$12 < \sigma^{(+)}(\alpha) < (2 \cdot 14)^2 = 28^2. \quad (51)$$

since $2 < \sigma^{(+)}(a_{ij}) < 14$ and $\sigma^{(+)}(2\cos(\pi/k)) \geq 1$ for $k \geq 3$.

Thus, α is a totally positive algebraic integer. By (50) and (51), the ground field $\mathbb{K} = \mathbb{Q}(\alpha)$ is generated by α , and $\mathbb{F}_{k,m} = \mathbb{Q}(\cos(2\pi/k), \cos(2\pi/m)) \subset \mathbb{K}$.

We get exactly the same situation which we had considered in [19, Sec. 5.5]. Only for the inequality (51) we had $< 14^2$ in [19] instead of 28^2 here. Then exactly the same considerations as in [19, Sec. 5.5] show that $[\mathbb{K} : \mathbb{Q}] \leq 56$. The worst case is achieved for $\{k, m\} = \{3, 113\}$.

More exactly, we apply the methods A and B of Case 2 in Sec. 3.1 to $a = 4$, $b_1 = 12$, $b_2 = 28^2$ (then $b = 28^2$) and $k := \max\{k, m\}$, $s := \min\{k, m\}$ where $k \geq s \geq 3$ and $s_0 = 3$.

At first, we apply the Method B. Since $\ln 2 > \ln(3)/2$, all k and s are non-exceptional, see (30). All exceptional pairs (k, s) that is when (32) which is

$$\ln 2 - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \leq 0 \quad (52)$$

satisfies, are $(k, s = 3)$ where $k = 3, 4, 5, 7, 8, 9, 11, 13, 17, 19$; $(k, s = 4)$ where $k = 4, 5$; $(k, s = 5)$ where $k = 5, 7$.

We can take $K_0 = 306$ in (36). Then $\Delta_1(4) = \ln(2) - \ln(3)/2 - \ln(307)/306 \geq 0.1251$, and $K_1 = 2760$ can be taken in (40). Checking (33) for $3 \leq s \leq k < 2760$, we obtain that $3 \leq s \leq 90$ and $3 \leq s \leq k \leq 420$. Moreover, $k \leq 90$ for $11 \leq s \leq 90$. For all these pairs (k, s) satisfying (33) which is

$$\frac{\varphi([k, s])}{2\rho(k, s)} \cdot \left(\ln 2 - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \right) < \ln 14 - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s}, \quad (53)$$

we obtain

$$[\mathbb{F}_{k,s} : \mathbb{Q}] = \frac{\varphi([k, s])}{2\rho(k, s)} \leq 56 \quad (54)$$

where 56 is achieved for $(k, s) = (113, 3)$. Moreover, for all these non-exceptional pairs (k, s) we obtain the bound (41) which is

$$N_0 = [\mathbb{K} : \mathbb{F}_{k,s}] \leq \left\lceil \frac{\ln 14 - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s}}{\frac{\varphi([k, s])}{2\rho(k, s)} \cdot \left(\ln 2 - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \right)} \right\rceil, \quad (55)$$

and finally we obtain the bound (42) which is

$$N = [\mathbb{K} : \mathbb{Q}] \leq \left\lceil \frac{\ln 14 - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s}}{\frac{\varphi([k, s])}{2\rho(k, s)} \cdot \left(\ln 2 - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \right)} \right\rceil \cdot \frac{\varphi([k, s])}{2\rho(k, s)}. \quad (56)$$

If either a pair (k, s) is exceptional, or the right hand side of (56) is more than 56 (these are possible only for pairs (k, s) with $3 \leq s \leq 7$ and $s \leq k \leq 420$), we also apply to the pair (k, s) the method A of the Case 2 to improve the poor bound (55) of $N_0 = [\mathbb{K} : \mathbb{F}_{k,s}]$ for non-exceptional (k, s) . We can apply this method to any pair (k, s) since (44) is valid for all $k \geq s \geq 3$ if $a = 4$. Finally we obtain that $[\mathbb{K} : \mathbb{Q}] \leq 56$.

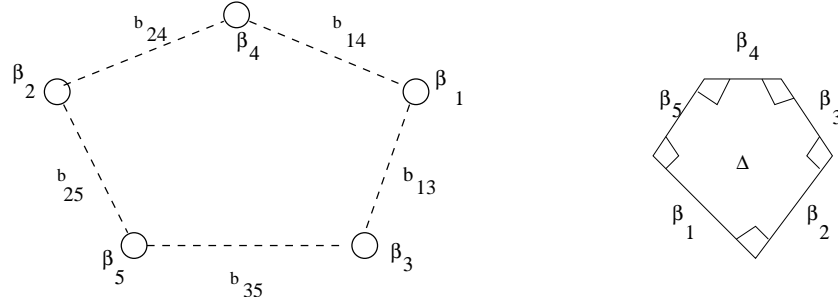


Figure 4: The graph of pentagon with right angles

3.3 V-arithmetic pentagon graphs $\Gamma_2^{(6)}(14)$, $\Gamma_3^{(6)}(14)$, $\Gamma_1^{(7)}(14)$, $\Gamma_2^{(7)}(14)$, and their ground fields

Here we consider V-arithmetic 3 and 4-dimensional graphs $\Gamma_2^{(6)}(14)$, $\Gamma_3^{(6)}(14)$, $\Gamma_1^{(7)}(14)$, $\Gamma_2^{(7)}(14)$ and their fields. See Definition 2.4 and Figure 2.

They are related to pentagons with right angles. This can be considered as a specialisation of the previous case when $c = 0$. See Figure 4.

The Gram matrix of any 4 elements β_i must have zero determinant. Considering this for all four elements β_i , we obtain equations

$$\begin{aligned} 4b_{14}^2 &= (4 - b_{13}^2)(4 - b_{24}^2), \\ 4b_{24}^2 &= (4 - b_{14}^2)(4 - b_{25}^2), \\ 4b_{25}^2 &= (4 - b_{24}^2)(4 - b_{35}^2), \\ 4b_{35}^2 &= (4 - b_{25}^2)(4 - b_{13}^2), \\ 4b_{13}^2 &= (4 - b_{35}^2)(4 - b_{14}^2). \end{aligned} \tag{57}$$

It follows that

$$\gamma = b_{13}b_{14}b_{24}b_{25}b_{35} = -(4 - b_{13}^2)(4 - b_{14}^2)(4 - b_{24}^2)(4 - b_{25}^2)(4 - b_{35}^2)/2^5 \tag{58}$$

since for the geometric embedding the sign must be positive. One can also see the sign from the determinant of the Gram matrix of β_1, \dots, β_5 which is equal to zero:

$$\begin{aligned} &8b_{13}^2 + 8b_{35}^2 + 8b_{25}^2 + 8b_{24}^2 + 8b_{14}^2 + 2b_{13}b_{14}b_{24}b_{25}b_{35} - \\ &- 2b_{13}^2b_{24}^2 - 2b_{13}^2b_{25}^2 - 2b_{14}^2b_{25}^2 - 2b_{14}^2b_{35}^2 - 2b_{24}^2b_{35}^2 - 32 = 0. \end{aligned}$$

Thus, (57) and (58) give equations of a hyperbolic pentagon with right angles.

For the geometric embedding $\sigma^{(+)}$ the expression γ is positive, and for $\sigma \neq \sigma^{(+)}$ it is negative because $0 < b_{ij}^2 < 4$ for all b_{ij} . We have the following very important statement.

Lemma 3.3. *The minimum of γ in (58) for b_{ij} satisfying (57) and $0 \leq b_{ij}^2 \leq 4$ is achieved for equal b_{ij}^2 . Then it is equal to*

$$-(\sqrt{5} - 1)^5 = -2.8854381999983 \dots$$

Proof. We denote $q_{ij} = 4 - a_{ij}^2$. Then equations (57) are

$$\begin{aligned} q_{14} &= 4 - \frac{q_{13}q_{24}}{4}, \\ q_{24} &= 4 - \frac{q_{14}q_{25}}{4}, \\ q_{25} &= 4 - \frac{q_{24}q_{35}}{4}, \\ q_{35} &= 4 - \frac{q_{25}q_{13}}{4}, \\ q_{13} &= 4 - \frac{q_{35}q_{14}}{4}. \end{aligned} \tag{59}$$

We should find maximum of

$$A = q_{13}q_{14}q_{24}q_{25}q_{35}$$

in the region $0 \leq q_{ij} \leq 4$ with the conditions (59). If one of $q_{ij} = 0$ or $q_{ij} = 4$, then $A = 0$. Thus, the maximum is taken when all $0 < q_{ij} < 4$.

We denote $x = q_{13}$ and $y = q_{24}$. Then $A = 2^6 F(x, y)$ where

$$F(x, y) = \frac{x^2y^2 + 16xy - 4x^2y - 4xy^2}{16 - xy},$$

and $0 < x < 4$ and $0 < y < 4$ are free variables. We have

$$\frac{\partial F}{\partial x} = \frac{-x^2y^3 + 4x^2y^2 + 32y^2x - 128xy - 64y^2 + 256y}{(16 - xy)^2}$$

and

$$\frac{\partial F}{\partial y} = \frac{-x^3y^2 + 4x^2y^2 + 32x^2y - 128xy - 64x^2 + 256x}{(16 - xy)^2}.$$

From $\frac{\partial F}{\partial x} = 0$, we get

$$-x^2y^2 + 4x^2y + 32yx - 128x - 64y + 256 = 0.$$

From $\frac{\partial F}{\partial y} = 0$, we get

$$-x^2y^2 + 4xy^2 + 32xy - 128y - 64x + 256 = 0.$$

Taking difference of these equations, we get $(x - y)(xy - 16) = 0$ which gives $x = y$ and $(x^2 + 4x - 16)(x - 4)^2 = 0$. It follows that $x = y = 2(\sqrt{5} - 1)$.

It follows the statement. \square

By Lemma 3.3, we have

$$-2.88543819\dots = -\gamma_0 = -(\sqrt{5} - 1)^5 < \sigma(\gamma) < 0. \tag{60}$$

where we denote

$$\gamma_0 = (\sqrt{5} - 1)^5 = 2.885438199983\dots \leq 2.8854382. \tag{61}$$

For pentagons Δ of graphs $\Gamma_2^{(6)}$, $\Gamma_3^{(6)}$, $\Gamma_1^{(7)}$ and $\Gamma_2^{(7)}$, we must replace β_i by δ_i or $\tilde{\delta}_i$ given in the definition which gives b_{ij} from a_{ij} . From these expressions, we get

$$b_{13}b_{14}b_{24}b_{25}b_{35} = \frac{a_{13}a_{14}a_{24}a_{25}a_{35}}{\sin^2 \frac{\pi}{m}} \quad \text{for } \Gamma_2^{(6)}, \quad (62)$$

$$b_{13}b_{14}b_{24}b_{25}b_{35} = \frac{a_{13}a_{14}a_{24}a_{25}a_{35} + 2 \cos \frac{\pi}{m_1} \cos \frac{\pi}{m_3} a_{14}a_{24}a_{25}a_{35}}{\sin^2 \frac{\pi}{m_1} \sin^2 \frac{\pi}{m_3}} \quad \text{for } \Gamma_3^{(6)}, \quad (63)$$

$$b_{13}b_{14}b_{24}b_{25}b_{35} = \frac{a_{13}a_{14}a_{24}a_{25}a_{35}}{\sin^2 \frac{\pi}{m_1} \sin^2 \frac{\pi}{m_2}} \quad \text{for } \Gamma_1^{(7)}, \quad (64)$$

$$b_{13}b_{14}b_{24}b_{25}b_{35} = \frac{a_{13}a_{14}a_{24}a_{25}a_{35} + 2 \cos \frac{\pi}{m_1} \cos \frac{\pi}{m_3} a_{14}a_{24}a_{25}a_{35}}{\sin^2 \frac{\pi}{m_1} \sin^2 \frac{\pi}{m_3} \sin^2 \frac{\pi}{m_2}} \quad \text{for } \Gamma_2^{(7)}. \quad (65)$$

We consider an algebraic integer $\alpha \in \mathbb{K}$ which is equal to

$$\alpha = \begin{cases} a_{13}a_{14}a_{24}a_{25}a_{35} & \text{for } \Gamma_2^{(6)} \text{ and } \Gamma_1^{(7)} \\ 2a_{13}a_{14}a_{24}a_{25}a_{35} + \\ + 4 \cos(\pi/m_1) \cos(\pi/m_3) a_{14}a_{24}a_{25}a_{35} & \text{for } \Gamma_3^{(6)} \text{ and } \Gamma_2^{(7)} \end{cases}. \quad (66)$$

By (60), for $\sigma \neq \sigma^{(+)}$, we obtain the inequalities

$$0 > \sigma(\alpha) > -\gamma_0 \sigma(\sin^2 \frac{\pi}{m}) \quad \text{for } \Gamma_2^{(6)}, \quad (67)$$

$$0 > \sigma(\alpha) > -2\gamma_0 \sigma(\sin^2 \frac{\pi}{m_1} \sin^2 \frac{\pi}{m_3}) \quad \text{for } \Gamma_3^{(6)}, \quad (68)$$

$$0 > \sigma(\alpha) > -\gamma_0 \sigma(\sin^2 \frac{\pi}{m_1} \sin^2 \frac{\pi}{m_2}) \quad \text{for } \Gamma_1^{(7)}, \quad (69)$$

$$0 > \sigma(\alpha) > -2\gamma_0 \sigma(\sin^2 \frac{\pi}{m_1} \sin^2 \frac{\pi}{m_3} \sin^2 \frac{\pi}{m_2}) \quad \text{for } \Gamma_2^{(7)}. \quad (70)$$

Since we consider 14-minimal graphs, we have

$$2^5 < \sigma^{(+)}(\alpha) < 14^5 \quad \text{for } \Gamma_2^{(6)}(14), \Gamma_1^{(7)}(14), \quad (71)$$

$$2^6 < \sigma^{(+)}(\alpha) < 2 \cdot 14^5 + 4 \cdot 14^4 = 32 \cdot 14^4 \quad \text{for } \Gamma_3^{(6)}(14), \Gamma_2^{(7)}(14), \quad (72)$$

It follows that $\mathbb{K} = \mathbb{Q}(\alpha)$. Since $2\gamma_0 = 5.7708763999663 \dots < 16$, all these cases are similar to considered in [19, Sec. 5.5] (and originally in [15]). We have adapted these considerations to our case here in Sec. 3.1, and we have applied them in Sec. 3.2. For each graph, we give details below which will be very important for further study.

3.3.1 V-arithmetic pentagon graphs $\Gamma_2^{(6)}(14)$.

For $\Gamma_2^{(6)}(14)$, considering $(-\alpha)$ from (66), we apply the methods A and B of Case 1 in Sec. 3.1 to $a = \gamma_0 = (\sqrt{5} - 1)^5 \leq 2.8854382$, $b_1 = -14^5$, $b_2 = -2^5$ (then $b = 14^5$), and $l := m$.

At first, we apply the Method B. All exceptional $l \geq 3$ that is when (12) which is

$$\ln \frac{2}{\sqrt{\gamma_0}} - \frac{\ln \gamma(l)}{\varphi(l)} \leq 0 \quad (73)$$

satisfies are $l = 3, 4, 5, 7, 8, 9, 11, 13, 17, 19$.

We can take $L_0 = 1540$ in (16). Then

$$\Delta_1(\gamma_0) = \ln\left(\frac{2}{\sqrt{\gamma_0}}\right) - \frac{\ln 1543}{1542} \geq 0.1585,$$

and $L_1 = 1595$ can be taken in (20). Checking (13) for $3 \leq l < 1595$, we obtain that $3 \leq l \leq 510$. For all these l such that (13) which is

$$\frac{\varphi(l)}{2} \cdot \left(\ln \frac{2}{\sqrt{\gamma_0}} - \frac{\ln \gamma(l)}{\varphi(l)} \right) < \ln \sqrt{\frac{14^5}{\gamma_0}} - \ln \sin \frac{\pi}{l} \quad (74)$$

satisfies, we obtain

$$[\mathbb{F}_l : \mathbb{Q}] = \frac{\varphi(l)}{2} \leq 75 \quad (75)$$

where 75 is achieved for $l = 151$. Moreover, for all these non-exceptional l we obtain the bound (21) which is

$$N_0 = [\mathbb{K} : \mathbb{F}_l] \leq \left\lceil \frac{\ln \sqrt{\frac{14^5}{\gamma_0}} - \ln \sin \frac{\pi}{l}}{\frac{\varphi(l)}{2} \cdot \left(\ln \frac{2}{\sqrt{\gamma_0}} - \frac{\ln \gamma(l)}{\varphi(l)} \right)} \right\rceil, \quad (76)$$

and finally we obtain the bound (22) which is

$$N = [\mathbb{K} : \mathbb{Q}] \leq \left\lceil \frac{\ln \sqrt{\frac{14^5}{\gamma_0}} - \ln \sin \frac{\pi}{l}}{\frac{\varphi(l)}{2} \cdot \left(\ln \frac{2}{\sqrt{\gamma_0}} - \frac{\ln \gamma(l)}{\varphi(l)} \right)} \right\rceil \cdot \frac{\varphi(l)}{2}. \quad (77)$$

If either l is exceptional, or the right hand side of (77) is more than 75 (this is possible only for $3 \leq l \leq 83$), we also apply to l the method A of the Case 1 to improve the poor bound (76) for $N_0 = [\mathbb{K} : \mathbb{F}_l]$ for non-exceptional l . We can apply this method to any $l \geq 3$ since (24) is valid for all $l \geq 3$ if $a = \gamma_0$. Finally we obtain that $[\mathbb{K} : \mathbb{Q}] \leq 75$.

3.3.2 V-arithmetic pentagon graphs $\Gamma_3^{(6)}(14)$.

In this case, considering $(-\alpha)$ from (66), we apply the methods A and B of Case 2 in Sec. 3.1 to $a = 2\gamma_0 \leq 5.7708764$, $b_1 = -32 \cdot 14^4$, $b_2 = -2^6$ (then $b = 32 \cdot 14^4$) and $k := \max\{m_1, m_3\}$, $s := \min\{m_1, m_3\}$ where $k \geq s \geq 3$ and $s_0 = 3$.

At first, we apply the Method B. Exceptional $l \geq 3$ satisfy (30) which is

$$\ln \frac{4}{\sqrt{2\gamma_0}} - \frac{\ln \gamma(l)}{\varphi(l)} \leq 0. \quad (78)$$

It follows that $l = 3$ is the only exceptional.

All exceptional pairs (k, s) where $k \geq s \geq 4$ that is when (32) which is

$$\ln \frac{4}{\sqrt{2\gamma_0}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \leq 0. \quad (79)$$

satisfies, are $(k, s = 4)$ where $k = 4, 5, 7, 8, 9, 11, 13, 17, 19$; $(k, s = 5)$ where $k = 5, 7, 8, 9, 11, 13, 17, 19, 23, 29, 31$; $(k, s = 7)$ where $k = 7, 11, 13$.

We can take $K_0 = 630$ in (36). Then

$$\Delta_1(2\gamma_0) = \ln \frac{4}{\sqrt{2\gamma_0}} - \frac{\ln 5}{4} - \frac{\ln 631}{630} \geq 0.097289,$$

and $K_1 = 4684$ can be taken in (40). Checking (33) for $4 \leq s \leq k < 4684$, we obtain that $4 \leq s \leq 210$ and $4 \leq s \leq k \leq 870$. Moreover, $k \leq 210$ for $14 \leq s \leq 210$. For all these pairs (k, s) satisfying (33) which is

$$\frac{\varphi([k, s])}{2\rho(k, s)} \cdot \left(\ln \frac{4}{\sqrt{2\gamma_0}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \right) < \ln \frac{4 \cdot 14^2}{\sqrt{\gamma_0}} - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s}, \quad (80)$$

we obtain

$$[\mathbb{F}_{k,s} : \mathbb{Q}] = \frac{\varphi([k, s])}{2\rho(k, s)} \leq 138 \quad (81)$$

where 138 is achieved for $(k, s) = (139, 5)$. Moreover, for all these non-exceptional pairs (k, s) we obtain the bound (41) which is

$$N_0 = [\mathbb{K} : \mathbb{F}_{k,s}] \leq \left[\frac{\ln \frac{4 \cdot 14^2}{\sqrt{\gamma_0}} - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s}}{\frac{\varphi([k, s])}{2\rho(k, s)} \cdot \left(\ln \frac{4}{\sqrt{2\gamma_0}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \right)} \right], \quad (82)$$

and finally we obtain the bound (42) which is

$$N = [\mathbb{K} : \mathbb{Q}] \leq \left[\frac{\ln \frac{4 \cdot 14^2}{\sqrt{\gamma_0}} - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s}}{\frac{\varphi([k, s])}{2\rho(k, s)} \cdot \left(\ln \frac{4}{\sqrt{2\gamma_0}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \right)} \right] \cdot \frac{\varphi([k, s])}{2\rho(k, s)}. \quad (83)$$

If either a pair (k, s) is exceptional, or the right hand side of (83) is more than 138 (these are possible only for pairs (k, s) with $4 \leq s \leq 11$ and $s \leq k \leq 870$), we also apply to the pair (k, s) the method A of the Case 2 to improve the poor bound (82) of $N_0 = [\mathbb{K} : \mathbb{F}_{k,s}]$ for non-exceptional (k, s) . We can apply this method to any pair (k, s) since (44) is valid for all $k \geq s \geq 3$ if $a = 2\gamma_0$. We obtain that $[\mathbb{K} : \mathbb{Q}] \leq 138$ for all $k \geq s \geq 4$.

Let us assume that $s = 3$ is exceptional. It means that either $m_1 = 3$ or $m_3 = 3$ for $\Gamma_3^{(6)}(14)$. For example, let $m_1 = 3$. Let us consider the V-arithmetic graph defined by e , δ_1 and δ_4 . We denote $\alpha = a_{14}^2$ where $a_{14} = \delta_1 \cdot \delta_4$. The determinant of the Gram matrix of e , δ_1 and δ_4 is equal to $-6 + 2\alpha$. It follows that $0 < \sigma(\alpha) < 3$ for $\sigma \neq \sigma^{(+)}$, and $4 < \sigma^{(+)}(\alpha) < 14^2$. Then $\mathbb{K} = \mathbb{Q}(\alpha)$. We apply Theorems 3.1 and 3.2 to $\mathbb{F} = \mathbb{Q}$ and α where $M = 1$, $B = 2$, $R = \sqrt{3}/2$, $S = 2 \cdot e \cdot 14^2/3$. We obtain $[\mathbb{K} : \mathbb{Q}] \leq 76$.

Thus, finally, $[\mathbb{K} : \mathbb{Q}] \leq 138$ for all graphs $\Gamma_3^{(6)}(14)$.

3.3.3 V-arithmetic pentagon graphs $\Gamma_1^{(7)}(14)$.

In this case, considering $(-\alpha)$ from (66), we apply the methods A and B of Case 2 in Sec. 3.1 to $a = \gamma_0 = (\sqrt{5} - 1)^5 \leq 2.8854382$, $b_1 = -14^5$, $b_2 = -2^5$ (then $b = 14^5$), and $k := \max\{m_1, m_2\}$, $s := \min\{m_1, m_2\}$ where $k \geq s \geq 3$ and $s_0 = 3$.

At first, we apply the Method B. Exceptional $l \geq 3$ satisfy (30) which is

$$\ln \frac{4}{\sqrt{\gamma_0}} - \frac{\ln \gamma(l)}{\varphi(l)} \leq 0. \quad (84)$$

It follows that all $l \geq 3$ are non-exceptional.

All exceptional pairs (k, s) where $k \geq s \geq 3$ that is when (32) which is

$$\ln \frac{4}{\sqrt{\gamma_0}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \leq 0 \quad (85)$$

satisfies, are $(k, s = 3)$ where $k = 3, 4, 5, 7$.

We can take $K_0 = 324$ in (36). Then

$$\Delta_1(2\gamma_0) = \ln \frac{4}{\sqrt{\gamma_0}} - \frac{\ln 3}{2} - \frac{\ln 331}{330} \geq 0.28956765,$$

and $K_1 = 1262$ can be taken in (40). Checking (33) for $3 \leq s \leq k < 1262$, we obtain that $3 \leq s \leq 90$ and $3 \leq s \leq k \leq 240$. Moreover, $k \leq 126$ for $6 \leq s \leq 90$. For all these pairs (k, s) satisfying (33) which is

$$\frac{\varphi([k, s])}{2\rho(k, s)} \cdot \left(\ln \frac{4}{\sqrt{\gamma_0}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \right) < \ln \sqrt{\frac{14^5}{\gamma_0}} - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s}, \quad (86)$$

we obtain

$$[\mathbb{F}_{k,s} : \mathbb{Q}] = \frac{\varphi([k, s])}{2\rho(k, s)} \leq 36 \quad (87)$$

where 36 is achieved for $(k, s) = (73, 3)$. Moreover, for all these non-exceptional pairs (k, s) we obtain the bound (41) which is

$$N_0 = [\mathbb{K} : \mathbb{F}_{k,s}] \leq \left[\frac{\ln \sqrt{\frac{14^5}{\gamma_0}} - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s}}{\frac{\varphi([k, s])}{2\rho(k, s)} \cdot \left(\ln \frac{4}{\sqrt{\gamma_0}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \right)} \right], \quad (88)$$

and finally we obtain the bound (42) which is

$$N = [\mathbb{K} : \mathbb{Q}] \leq \left[\frac{\ln \sqrt{\frac{14^5}{\gamma_0}} - \ln \sin \frac{\pi}{k} - \ln \sin \frac{\pi}{s}}{\frac{\varphi([k, s])}{2\rho(k, s)} \cdot \left(\ln \frac{4}{\sqrt{\gamma_0}} - \frac{\ln \gamma(k)}{\varphi(k)} - \frac{\ln \gamma(s)}{\varphi(s)} \right)} \right] \cdot \frac{\varphi([k, s])}{2\rho(k, s)}. \quad (89)$$

If either a pair (k, s) is exceptional, or the right hand side of (89) is more than 36 (these are possible only for pairs (k, s) with $3 \leq s \leq 5$ and $s \leq k \leq 240$), we also apply to the pair (k, s) the method A of the Case 2 to improve the poor bound (88) of $N_0 = [\mathbb{K} : \mathbb{F}_{k, s}]$ for non-exceptional (k, s) . We can apply this method to any pair (k, s) since (44) is valid for all $k \geq s \geq 3$ if $a = \gamma_0$.

For all $3 \leq s \leq 5$ and $s \leq k \leq 240$, the Method A gives $[\mathbb{K} : \mathbb{Q}] \leq 36$ except $s = k = 3$ (equivalently, $m_1 = m_2 = 3$). Finally $[\mathbb{K} : \mathbb{Q}] \leq 42$ for all graphs $\Gamma_1^{(7)}(14)$.

3.3.4 V-arithmetic pentagon graphs $\Gamma_2^{(7)}(14)$.

Since $0 < \sigma(\sin^2(\pi/m_2)) < 1$ for any embedding $\sigma : \mathbb{K} \rightarrow \mathbb{R}$, in this case, we have for α from (66) the same inequalities (68) and (72) as for $\Gamma_3^{(6)}(14)$. Thus, we get the same upper bound $[\mathbb{K} : \mathbb{Q}] \leq 138$ as for $\Gamma_3^{(6)}(14)$.

This finishes the proof of Theorem 2.5.

4 Appendix: Some results about cyclotomic fields

Here we give some results about cyclotomic fields which we used. All of them follow from standard results. For example, see the book [5].

We consider the cyclotomic field $\mathbb{Q}(\sqrt[l]{1})$ and its totally real subfield $\mathbb{F}_l = \mathbb{Q}(\cos(2\pi/l))$. We have $[\mathbb{Q}(\sqrt[l]{1}) : \mathbb{Q}] = \varphi(l)$ where $\varphi(l)$ is the Euler function. We have $\mathbb{F}_l = \mathbb{Q}(\sqrt[l]{1}) = \mathbb{Q}$ for $l = 1, 2$, and $[\mathbb{F}_l : \mathbb{Q}] = \varphi(l)/2$ for $l \geq 3$. It is known (e.g., see [5]) that the discriminant of the field $\mathbb{Q}(\sqrt[l]{1})$ is equal to (where p is prime)

$$|\text{discr } \mathbb{Q}(\sqrt[l]{1})| = \frac{l^{\varphi(l)}}{\prod_{p|l} p^{\varphi(l)/(p-1)}}. \quad (90)$$

Let $\zeta_l = \exp(2\pi i/l)$ be a primitive l -th root of 1. The element ζ_l generates the ring of integers of $\mathbb{Q}(\sqrt[l]{1})$. Further we assume that $l \geq 3$. The equation of ζ_l over \mathbb{F}_l is $g(x) = (x - \zeta_l)(x - \zeta_l^{-1}) = x^2 - (\zeta_l + \zeta_l^{-1})x + 1 = 0$. We have $g'(\zeta_l) = 2\zeta_l - (\zeta_l + \zeta_l^{-1}) = \zeta_l - \zeta_l^{-1}$. Thus,

$$N_{\mathbb{Q}(\sqrt[l]{1})/\mathbb{F}_l}(g'(\zeta_l)) = (\zeta_l - \zeta_l^{-1})(\zeta_l^{-1} - \zeta_l) = 4\sin^2(2\pi/l)$$

which gives the discriminant $\delta(\mathbb{Q}(\sqrt[l]{1})/\mathbb{F}_l) = 4\sin^2(2\pi/l)$. It follows

$$|\delta(\mathbb{Q}(\sqrt[l]{1})/\mathbb{Q})| = |\delta(\mathbb{F}_l/\mathbb{Q})^2 N_{\mathbb{F}_l/\mathbb{Q}}(4\sin^2(2\pi/l))|.$$

We have

$$N_{\mathbb{F}_l/\mathbb{Q}}(4 \sin^2(\pi/l)) = \gamma(l) = \begin{cases} p & \text{if } l = p^t > 2 \text{ where } p \text{ is prime,} \\ 1 & \text{otherwise.} \end{cases} \quad (91)$$

If l is odd, then $4 \sin^2(\pi/l)$ and $4 \sin^2(2\pi/l)$ are conjugate, and their norms are equal. Thus,

$$N_{\mathbb{F}_l/\mathbb{Q}}(4 \sin^2(2\pi/l)) = \gamma(l), \text{ if } l \geq 3 \text{ is odd.}$$

If l is even and $l_1 = l/2$, then $4 \sin^2(2\pi/l) = 4 \sin^2(\pi/l_1)$. If l_1 is odd, then $\mathbb{F}_{l_1} = \mathbb{F}_l$, and we get

$$N_{\mathbb{F}_l/\mathbb{Q}}(4 \sin^2(2\pi/l)) = \gamma(l/2) \text{ if } l \geq 6 \text{ is even, but } l/2 \text{ is odd.}$$

If $l_1 \geq 4$ is even, then $[\mathbb{F}_l : \mathbb{F}_{l_1}] = 2$, and we get

$$N_{\mathbb{F}_l/\mathbb{Q}}(4 \sin^2(2\pi/l)) = \gamma(l/2)^2 \text{ if } l \geq 8 \text{ and } l/2 \text{ is even.}$$

At last,

$$N_{\mathbb{F}_4/\mathbb{Q}}(4 \sin^2(2\pi/4)) = 4$$

if $l = 4$.

Thus, finally we get for $l \geq 3$:

$$N_{\mathbb{F}_l/\mathbb{Q}}(4 \sin^2(2\pi/l)) = \tilde{\gamma}(l) = \begin{cases} \gamma(l) & \text{if } l \geq 3 \text{ is odd,} \\ \gamma(l/2) & \text{if } l/2 \geq 3 \text{ is odd,} \\ \gamma(l/2)^2 & \text{if } l/2 \geq 4 \text{ is even,} \\ 4 & \text{if } l = 4. \end{cases} \quad (92)$$

Moreover, we obtain the formula for the discriminant:

$$|\text{discr } \mathbb{F}_l| = \left(|\text{discr } \mathbb{Q}(\sqrt[l]{1})| / \tilde{\gamma}(l) \right)^{1/2} \text{ for } l \geq 3 \quad (93)$$

where $|\text{discr } \mathbb{Q}(\sqrt[l]{1})|$ is given by (90), and $\tilde{\gamma}(l)$ is given by (92).

We denote $\mathbb{F}_{k,s} = \mathbb{Q}(\cos(2\pi/k), \cos(2\pi/s))$. Further we assume that $k, s \geq 3$. Let $m = [k, s]$ be the least common multiple of k and s . Then $\mathbb{F}_{k,s} \subset \mathbb{F}_m \subset \mathbb{Q}(\sqrt[m]{1})$. We have $\text{Gal}(\mathbb{Q}(\sqrt[m]{1})/\mathbb{Q}) = (\mathbb{Z}/m\mathbb{Z})^*$ where $\alpha \in (\mathbb{Z}/m\mathbb{Z})^*$ acts on each m -th root ζ of 1 by the formula $\zeta \mapsto \zeta^\alpha$. Obviously, $\mathbb{F}_{k,s}$ is the fixed field of the subgroup G of the Galois group $(\mathbb{Z}/m\mathbb{Z})^*$ which consists of all $\alpha \in (\mathbb{Z}/m\mathbb{Z})^*$ such that $\alpha \equiv \pm 1 \pmod{k}$ and $\alpha \equiv \pm 1 \pmod{s}$. The G includes the subgroup of order two of $\alpha \equiv \pm 1 \pmod{m}$. If $\alpha \equiv 1 \pmod{k}$, then $\alpha \equiv 1 + kt \pmod{m}$ where $t \in \mathbb{Z}$. If $1 + kt \equiv -1 \pmod{s}$, then the equation $kt + sr = 2$ has an integer solution (t, r) which is equivalent to $(k, s)|2$. Thus, G has the order 4 if and only if $(k, s)|2$. Otherwise, G has the order 2. We set

$$\rho(k, s) = \begin{cases} 2 & \text{if } (k, s)|2, \\ 1 & \text{otherwise,} \end{cases} \quad (94)$$

and we obtain

$$[\mathbb{F}_{k,s} : \mathbb{Q}] = \frac{\varphi(m)}{2\rho(k,s)}. \quad (95)$$

Moreover, we get

$$\mathbb{F}_{k,s} = \mathbb{F}_m \text{ if } (k,s) \nmid 2.$$

It follows,

$$|\text{discr } \mathbb{F}_{k,s}| = |\text{discr } \mathbb{F}_m| \text{ if } (k,s) \nmid 2, \quad (96)$$

where $m = [k, s]$, and $|\text{discr } \mathbb{F}_m|$ is given by (93).

Assume that $(k,s) \mid 2$. If $(k,s) = 1$, then the fields $\mathbb{Q}(\sqrt[k]{1})$ and $\mathbb{Q}(\sqrt[s]{1})$ are linearly disjoint and their discriminants are coprime. Then their subfields \mathbb{F}_k and \mathbb{F}_s are linearly disjoint, and their discriminants are coprime, and we obtain

$$|\text{discr } \mathbb{F}_{k,s}| = |\text{discr } \mathbb{F}_k|^{(\varphi(s)/2)} |\text{discr } \mathbb{F}_s|^{(\varphi(k)/2)} \text{ if } (k,s) = 1 \text{ and } k, s \geq 3.$$

Assume that $(k,s) = 2$. Then one of $k/2$ or $s/2$ is odd. Assume, $k_1 = k/2$ is odd. Then $\mathbb{F}_k = \mathbb{F}_{k_1}$ and $\mathbb{F}_{k,s} = \mathbb{F}_{k_1,s}$ where $(k_1,s) = 1$. Thus, we obtain the previous case which gives exactly the same formula. We finally obtain

$$|\text{discr } \mathbb{F}_{k,s}| = |\text{discr } \mathbb{F}_k|^{(\varphi(s)/2)} |\text{discr } \mathbb{F}_s|^{(\varphi(k)/2)} \text{ if } (k,s) \mid 2 \text{ and } k, s \geq 3 \quad (97)$$

where the discriminants $|\text{discr } \mathbb{F}_k|$ and $|\text{discr } \mathbb{F}_s|$ are given by (93).

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